Objective Lagrangian Vortex Cores and their Visual Representations Additional Material

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1 Setup of the Linear System

In the main paper, we have shown that the search for the optimal observer rotation along pathlines requires minimizing:

$$\widehat{e}_{\mathbf{p}_0,t_0,\tau} = g - 2\,\mathbf{u}^{\mathrm{T}}\mathbf{c} + \mathbf{c}^{\mathrm{T}}\mathbf{M}\,\mathbf{c} \tag{1}$$

which is quadratic in the unknowns $\mathbf{c}$. The coefficients $g,\mathbf{u},\mathbf{M}$ are computed in 2D in the following way:

$$g = \frac{1}{N+1}\sum_{i=0}^{N}\overline{g}_i \tag{2}$$

$$\mathbf{u} = \frac{1}{N+1}(\mathbf{u}_1 + \mathbf{L}^{\mathrm{T}}\mathbf{u}_2) \tag{3}$$

$$\mathbf{M} = \frac{1}{N+1}(\mathbf{M}_{1,1} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{2,2}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{1,2} + \mathbf{M}_{1,2}\mathbf{L}) \tag{4}$$

with $\mathbf{u}_i = (\overline{\mathbf{u}}_0[i],\ldots,\overline{\mathbf{u}}_N[i])^{\mathrm{T}}$ and $\mathbf{M}_{i,j} = \mathrm{diag}((\overline{\mathbf{M}}_0[i,j],\ldots,\overline{\mathbf{M}}_N[i,j])^{\mathrm{T}})$ is a diagonal matrix. In 3D, we get for $\mathbf{u}$ and $\mathbf{M}$ instead:

$$\mathbf{u} = \frac{1}{N+1}\begin{pmatrix}\mathbf{u}_1+\mathbf{L}^{\mathrm{T}}\mathbf{u}_4\\ \mathbf{u}_2+\mathbf{L}^{\mathrm{T}}\mathbf{u}_5\\ \mathbf{u}_3+\mathbf{L}^{\mathrm{T}}\mathbf{u}_6\end{pmatrix} \tag{5}$$

$$\mathbf{M} = \frac{1}{N+1}\begin{pmatrix}\mathbf{H}_{11} & \mathbf{H}_{12} & \mathbf{H}_{13}\\ \mathbf{H}_{12}{}^{\mathrm{T}} & \mathbf{H}_{22} & \mathbf{H}_{23}\\ \mathbf{H}_{13}{}^{\mathrm{T}} & \mathbf{H}_{23}{}^{\mathrm{T}} & \mathbf{H}_{33}\end{pmatrix} \tag{6}$$

with the auxiliary matrices:

$$\begin{aligned}
\mathbf{H}_{11} &= (\mathbf{M}_{1,1} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{4,4}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{1,4} + \mathbf{M}_{1,4}\mathbf{L})\\
\mathbf{H}_{12} &= (\mathbf{M}_{1,2} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{4,5}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{2,4} + \mathbf{M}_{1,5}\mathbf{L})\\
\mathbf{H}_{13} &= (\mathbf{M}_{1,3} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{4,6}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{3,4} + \mathbf{M}_{1,6}\mathbf{L})\\
\mathbf{H}_{22} &= (\mathbf{M}_{2,2} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{5,5}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{2,5} + \mathbf{M}_{2,5}\mathbf{L})\\
\mathbf{H}_{23} &= (\mathbf{M}_{2,3} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{5,6}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{3,5} + \mathbf{M}_{2,6}\mathbf{L})\\
\mathbf{H}_{33} &= (\mathbf{M}_{3,3} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{6,6}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{M}_{3,6} + \mathbf{M}_{3,6}\mathbf{L})
\end{aligned}$$

The equivalence of $\widehat{e}_{\mathbf{p}_0,t_0,\tau}$ and Eqs. (1)–(6) is shown below in Section 4. Finally, we used second-order accurate central differences for estimating the derivatives of ω using $\Delta t = \frac{\tau}{N}$, i.e.,

$$\mathbf{L} = \frac{1}{2\,\Delta t}\begin{pmatrix}
-3 & 4 & -1 & 0 & \ldots & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
0 & 0 & 0 & 0 & \ldots & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 1\\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & -4 & 3
\end{pmatrix} \tag{7}$$

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As shown in the main paper, $\mathbf{u}$ and $\mathbf{M}$ are then used to solve for the optimal observer rotation, which is stored in $\mathbf{c}_{opt}$:

$$\mathbf{c}_{opt} = \mathbf{M}^{-1}\mathbf{u}. \tag{8}$$

2 Proof of Lemma 1

To prove Lemma 1, we have to show

$$\begin{aligned}
e(\mathbf{x}_{Ra},\mathbf{a}_{Ra},t) &= 0 && (9)\\
\nabla_{\mathbf{xa}}e(\mathbf{x}_{Ra},\mathbf{a}_{Ra},t) &= \mathbf{0} && (10)\\
\mathbf{H}_{\mathbf{xa}}(\mathbf{x}_{Ra},\mathbf{a}_{Ra},t) &\ \text{is}\ \text{positive definite} && (11)
\end{aligned}$$

where $\nabla_{\mathbf{xa}}e = \left(\frac{\partial e}{\partial x},\frac{\partial e}{\partial y},\frac{\partial e}{\partial a},\frac{\partial e}{\partial b},\frac{\partial e}{\partial \omega}\right)^{\mathrm{T}}$ is the gradient of e in both the space and the parameters of the Killing field, and $\mathbf{H}_{\mathbf{xa}} = \nabla_{\mathbf{xa}}(\nabla_{\mathbf{xa}}e)$ is the Hessian matrix of e. Eqs. (9) and (10) are easy to see by inserting $\mathbf{x}_{Ra}$ and $\mathbf{a}_{Ra}$ and evaluating. To show Eq. (11), we decompose $\mathbf{H}_{\mathbf{xa}}$ into 3 components $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_R$ by

$$\mathbf{H}_{\mathbf{xa}}(\mathbf{x}_{Ra},\mathbf{a}_{Ra},t) = \mu\,\mathbf{H}_1 + \mu\,R^2\,\mathbf{H}_R + \mathbf{H}_2 \tag{12}$$

with

$$\mathbf{H}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 8 & 0 & 240\,(t-1/2)\,s^2\\
0 & 0 & 0 & 32 & 640\,s^3\\
0 & 0 & 240\,(t-1/2)\,s^2 & 640\,s^3 & 200\,s^4\,(64\,s^2-36\,s+9)
\end{pmatrix} \tag{13}$$

$$\mathbf{H}_R = \begin{pmatrix}
0&0&0&0&0\\
0&0&0&0&0\\
0&0&0&0&0\\
0&0&0&0&0\\
0&0&0&0&4
\end{pmatrix} \tag{14}$$

$$\mathbf{H}_2 = \begin{pmatrix}
8 & 0 & 0 & -4 & h_{1,5}\\
0 & 32 & 8 & 0 & h_{2,5}\\
0 & 8 & 2 & 0 & h_{3,5}\\
-4 & 0 & 0 & 2 & h_{4,5}\\
h_{1,5} & h_{2,5} & h_{3,5} & h_{4,5} & h_{5,5}
\end{pmatrix} \tag{15}$$

with

$$\begin{aligned}
h_{1,5} &= -80\,s^3 && (16)\\
h_{2,5} &= -240\,(t-1/2)\,s^2 && (17)\\
h_{3,5} &= -60\,(t-1/2)\,s^2 && (18)\\
h_{4,5} &= 40\,s^3 && (19)\\
h_{5,5} &= 50\left(16\,s^2-36\,s+9\right)s^4. && (20)
\end{aligned}$$

For them, it holds

$$\text{Rank}(\mathbf{H}_1) = \text{Rank}(\mathbf{H}_2) = 2 \quad , \quad \text{Rank}(\mathbf{H}_R) = 1. \tag{21}$$

We show that $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_R$ are positive semi-definite in the following way: let s_1, s_2 be the sum of the two non-zero eigenvalues of $\mathbf{H}_1, \mathbf{H}_2$, respectively. Further, let p_1, p_2 be the product of the two non-zero eigenvalues of $\mathbf{H}_1, \mathbf{H}_2$, respectively. This gives

$$s_1 = 200\,s^4\,(-4\,s\,(-16\,s+9)+9)+40 \tag{22}$$
$$p_1 = 6400\,s^4\,(-4\,s\,(-4\,s+9)+9)+256 \tag{23}$$
$$s_2 = 50\,s^4\,(-4\,s\,(-4\,s+9)+9)+44 \tag{24}$$
$$p_2 = 100\,s^4\,(-4\,s\,(-68\,s+45)+45)+340 \tag{25}$$

Keeping in mind $0 \leq s \leq \frac{1}{4}$, we get

$$s_1, s_2, p_1, p_2 > 0 \tag{26}$$

which shows that $\mathbf{H}_1, \mathbf{H}_2$ are positive semi-definite. The positive semi-definiteness of $\mathbf{H}_R$ is obvious, which gives that $\mathbf{H}_{\mathbf{xa}}$ in (12) is positive semi-definite as well. To show that $\mathbf{H}_{\mathbf{xa}}$ is positive definite, we have to additionally show that $\mathbf{H}_{\mathbf{xa}}$ has full rank. This is done by computing

$$\det(\mathbf{H}_{\mathbf{xa}}(\mathbf{x}_{Ra}, \mathbf{a}_{Ra}, t)) = 262144\,R^2\,\mu^3 \tag{27}$$

which gives that $\mathbf{H}_{\mathbf{xa}}$ is positive definite for positive μ, R. The sheet "ProofLemma1.txt" in the additional material presents a Maple proof of this.

3 PROOF OF $\mathbf{m}_\mathbf{p}$

To proof that $\mathbf{m}_\mathbf{p}$ in Eq. (34) and $\mathbf{m}_{r,\mathbf{p}}$ in Eq. (35) of the main paper are identical to Eqs. (43)-(51) of the main paper is shown in the accompanying Maple sheets "Proofmp2D.txt" and "Proofmp3D.txt" for both 2D and 3D.

4 PROOF OF $\widehat{e}_{\mathbf{p}_0,t_0,\tau}$

The equivalence of Eq. (62) of the main paper and Eqs. (1)–(6) of this additional material is shown in the accompanying Maple sheets "Proofehat2D.txt" and "Proofehat3D.txt" for both 2D and 3D.