Unified Smooth Vector Graphics: Modeling Gradient Meshes and Curve-based Approaches Jointly as Poisson Problem – Supplemental Material

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I. Laplacian of Gradient Mesh

Gradient meshes consist of one or multiple Ferguson patches, each containing bi-cubic color patches, whose Laplacian is not necessarily homogeneous. In fact, the ability to add more variation to the color gradient is what makes gradient meshes artistically expressive. In the following, we let $\mathbf{u} = (u, v) \in [0,1]^2$ be the UV coordinate that parameterizes a single Ferguson patch with colors $\mathbf{c}(\mathbf{u})$ and 2D spatial coordinates $\mathbf{x}(\mathbf{u})$, following Eq. (1), cf. Sun et al. [2]:

$$f(u,v) = \mathbf{t}(u)^{\mathrm{T}} \cdot \mathbf{C}^{\mathrm{T}} \cdot \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{t}(v) \tag{1}$$

with the monomial basis $\mathbf{t}(t) = (1, t, t^2, t^3)^{\mathrm{T}}$ and the matrices:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} f_{0,0} & f_{0,3} & \partial_v f_{0,0} & \partial_v f_{0,3} \\ f_{3,0} & f_{3,3} & \partial_v f_{3,0} & \partial_v f_{3,3} \\ \partial_u f_{0,0} & \partial_u f_{0,3} & \partial_u \partial_v f_{0,0} & \partial_u \partial_v f_{0,3} \\ \partial_u f_{3,0} & \partial_u f_{3,3} & \partial_u \partial_v f_{3,0} & \partial_u \partial_v f_{3,3} \end{pmatrix}$$

Since a Ferguson patch is parameterized in UV coordinates $\mathbf{u}$ and since the Poisson equation

$$\Delta \mathbf{c}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad s.t.\ \mathcal{B} \in \partial\Omega \tag{2}$$

is in spatial coordinates $\mathbf{x}$, coordinate transformations are needed, including the mapping from $\mathbf{x}$ to $\mathbf{u}$ and its coordinate partials, as well as the gradient and the Laplacian of color with respect to $\mathbf{x}$, as explained in the following.

a) Pre-Image: The pre-image from image space coordinates $\mathbf{x}$ back to $\mathbf{u}$ is later for simplicity referred to as $\mathbf{u}(\mathbf{x})$. The pre-image exists if the coordinates $\mathbf{x}(\mathbf{u})$ do not contain folds, i.e., if $\mathbf{x}(\mathbf{u})$ is a homeomorphism. Inside that function, we find the $\mathbf{u}_0$ coordinate that maps to image space coordinate $\mathbf{x}_0$ by solving the root finding problem $\mathbf{x}(\mathbf{u}_0) - \mathbf{x}_0 = \mathbf{0}$, which is done using Bézier clipping [1]. Thus, given an image space coordinate $\mathbf{x}$, we can evaluate the color of a Ferguson patch at the corresponding UV coordinate via $\mathbf{c}(\mathbf{u}(\mathbf{x}))$.

b) Coordinate Jacobian: The partial derivatives of the inverse coordinate transformation are for $\mathbf{x}(\mathbf{u}) = (x(u,v), y(u,v))^{\mathrm{T}}$:

$$\begin{pmatrix} \frac{\partial}{\partial x}u & \frac{\partial}{\partial x}v & 0 & 0 & 0 \\ \frac{\partial}{\partial y}u & \frac{\partial}{\partial y}v & 0 & 0 & 0 \\ \frac{\partial^2}{\partial x^2}u & \frac{\partial^2}{\partial x^2}v & \frac{\partial}{\partial x}u\frac{\partial}{\partial x}u & \frac{\partial}{\partial x}v\frac{\partial}{\partial x}v & 2\frac{\partial}{\partial x}u\frac{\partial}{\partial x}v \\ \frac{\partial^2}{\partial y^2}u & \frac{\partial^2}{\partial y^2}v & \frac{\partial}{\partial y}u\frac{\partial}{\partial y}u & \frac{\partial}{\partial y}v\frac{\partial}{\partial y}v & 2\frac{\partial}{\partial y}u\frac{\partial}{\partial y}v \\ \frac{\partial^2}{\partial x \partial y}u & \frac{\partial^2}{\partial x \partial y}v & \frac{\partial}{\partial x}u\frac{\partial}{\partial y}u & \frac{\partial}{\partial x}v\frac{\partial}{\partial y}v & \frac{\partial}{\partial x}u\frac{\partial}{\partial y}v + \frac{\partial}{\partial y}u\frac{\partial}{\partial x}v \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} \frac{\partial}{\partial u}x & \frac{\partial}{\partial u}y & 0 & 0 & 0 \\ \frac{\partial}{\partial y}x & \frac{\partial}{\partial y}y & 0 & 0 & 0 \\ \frac{\partial^2}{\partial u^2}x & \frac{\partial^2}{\partial u^2}y & \frac{\partial}{\partial u}x\frac{\partial}{\partial u}x & \frac{\partial}{\partial u}y\frac{\partial}{\partial u}y & 2\frac{\partial}{\partial u}x\frac{\partial}{\partial u}y \\ \frac{\partial^2}{\partial v^2}x & \frac{\partial^2}{\partial v^2}y & \frac{\partial}{\partial v}x\frac{\partial}{\partial v}x & \frac{\partial}{\partial v}y\frac{\partial}{\partial v}y & 2\frac{\partial}{\partial v}x\frac{\partial}{\partial v}y \\ \frac{\partial^2}{\partial u \partial v}x & \frac{\partial^2}{\partial u \partial v}y & \frac{\partial}{\partial u}x\frac{\partial}{\partial v}x & \frac{\partial}{\partial u}y\frac{\partial}{\partial v}y & \frac{\partial}{\partial u}x\frac{\partial}{\partial v}y + \frac{\partial}{\partial v}x\frac{\partial}{\partial u}y \end{pmatrix}^{-1}$$

c) Gradient: The spatial color gradient is by chain rule:

$$\frac{\partial \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{x}} = \frac{\partial c(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{u}} \frac{\partial \mathbf{u}(\mathbf{x})}{\partial \mathbf{x}}. \tag{4}$$

d) Laplacian: The Poisson problem in Eq. (2) requires the Laplacian of color with respect to image space $\Delta_{\mathbf{x}}\mathbf{c}(\mathbf{u}(\mathbf{x}))$, which is likewise computed via chain rule:

$$\begin{aligned} \Delta_{\mathbf{x}}\mathbf{c}(\mathbf{u}(\mathbf{x})) &= \frac{\partial^2 \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial x^2} + \frac{\partial^2 \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial y^2} \\ &= \frac{\partial \mathbf{u}(\mathbf{x})}{\partial x}^{\mathrm{T}} \frac{\partial^2 \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{u}^2} \frac{\partial \mathbf{u}(\mathbf{x})}{\partial x} + \frac{\partial \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{u}}^{\mathrm{T}} \frac{\partial^2 \mathbf{u}}{\partial x^2} \\ &+ \frac{\partial \mathbf{u}(\mathbf{x})}{\partial y}^{\mathrm{T}} \frac{\partial^2 \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{u}^2} \frac{\partial \mathbf{u}(\mathbf{x})}{\partial y} + \frac{\partial \mathbf{c}(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{u}}^{\mathrm{T}} \frac{\partial^2 \mathbf{u}}{\partial y^2}. \end{aligned} \tag{5}$$

Using the above ingredients, we can now add the Laplacian of each Ferguson patch of a gradient mesh $\Delta_{\mathbf{x}}\mathbf{c}(\mathbf{u}(\mathbf{x}))$ to the patch Laplacian $\mathbf{f}(\mathbf{x})$:

$$\mathbf{f}(\mathbf{x}) = \underbrace{\sum_{\mathbf{p}(t) \in \mathcal{P}} \mathbf{f}_{\mathbf{p}(t)}(\mathbf{x})}_{\text{Poisson curves}} + \underbrace{\sum_{\mathbf{c}_i(\mathbf{u}) \in \mathcal{G}} \lambda_i \cdot \Delta_{\mathbf{x}}\mathbf{c}_i(\mathbf{u}(\mathbf{x}))}_{\text{Gradient meshes}}. \tag{6}$$

The symbolic derivatives as described above are an alternative to the numerical computation via finite differences from a rasterized gradient mesh image. Unlike symbolic derivatives, the error of the numerical computation depends on the image resolution, which is likewise the case when the PDE solver estimates the derivatives with finite differences. The symbolic derivatives could also be useful for differentiable rasterizers that might be employed in the future for automatic vectorization.

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